

Stochastic continuity equation with non-smooth velocity.

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Abstract

In this article we study the existence and uniqueness of solutions of stochastic continuity equation with irregular coefficients.

1 Introduction

Several physical phenomena arising in fluid dynamics and kinetic equations can be modeled by the continuity/ transport equation,

$$\partial_t u(t, x) + \operatorname{div}(b(t, x)u(t, x)) = 0, \quad (1.1)$$

where u is the the physical quantity that evolves in time. Such quantities are the vorticity of a fluid, or the density of a collection of particles advected by a velocity field which is highly irregular, in the sense that it has a derivative given by a distribution and a nonlinear dependence on the solution u . For application in the fluid dynamics see Lions' books [18],[19] and for applications in the domain of conservation laws see Dafermos' book [5].

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Recently research activity has been devoted to study continuity equations with rough coefficients, showing that a well-posedness result. The main motivation comes from the study of some nonlinear partial differential equations of the mathematical physics.

We put focus in uniqueness issue, Di Perna and Lions [7] have introduced the notion of renormalized solution to this equation: it is a solution such that

$$\partial_t \beta(u(t, x)) + \operatorname{div}(b(t, x) \cdot \beta(u(t, x))) = 0. \quad (1.2)$$

for any suitable non-linearity β . Notice that (1.2) holds for smooth solutions, by an immediate application of the chain-rule. The renormalization property asserts that nonlinear compositions of the solution are again solutions, or alternatively that the chain-rule holds in this weak context. The overall result which motivates this definition is that, if the renormalization property holds, then solutions of (1.1) are unique and stable.

In the case that b has $W^{1,1}$ spatial regularity (together with a condition of boundedness on the divergence) the commutator lemma between smoothing convolution and weak solution can be proved and, as a consequence, all L^∞ -weak solutions are renormalized. The theory has been generalized by L. Ambrosio [1] to the case of only BV regularity for b instead of $W^{1,1}$. In the case of two-dimensional vector-field, we also refer to the work of F. Bouchut and L. Desvillettes [4] in which the case of divergence free vector-field with continuous coefficient is treated, and in [12] in which this result is extended to vector-field with L^2_{loc} coefficients with a condition of regularity on the direction of the vector-field. We refer the readers to the two excellent summaries in [2] and [6].

In recent years, much attention has been devoted to extensions of this theory under random perturbations of the drift vector field, namely considering the following stochastic linear transport/continuity equation

$$\begin{cases} \partial_t u(t, x) + \operatorname{Div}\left((b(t, x) + \frac{dB_t}{dt}) \cdot u(t, x)\right) = 0, \\ u|_{t=0} = u_0. \end{cases} \quad (1.3)$$

Here, $(t, x) \in [0, T] \times \mathbb{R}^d$, $\omega \in \Omega$ is an element of the probability space $(\Omega, \mathbb{P}, \mathcal{F})$, $b : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a given vector field and $B_t = (B_t^1, \dots, B_t^d)$

is a standard Brownian motion in \mathbb{R}^d . The stochastic integration is to be understood in the Stratonovich sense.

A very interesting situation is when the stochastic problem is better behaved than the deterministic one. A first result in this direction was given by F. Flandoli, M. Gubinelli and E. Priola in [11], where they obtained well-posedness of the stochastic problem for an Hölder continuous drift term, with some integrability conditions on the divergence. Their approach is based on a careful analysis of the characteristics. Using a similar approach, E. Fedrizzi and F. Flandoli in [8] obtained a well-posedness result in the class $W_{loc}^{1,p}$ -solution under only some integrability conditions on the drift, with no assumption on the divergence, but for fairly regular initial conditions. There, it is only assumed that

$$\begin{aligned} b &\in L^q([0, T]; L^p(\mathbb{R}^d)) , \\ \text{for } \quad p, q &\in [2, \infty) , \quad \frac{d}{p} + \frac{2}{q} < 1 . \end{aligned} \tag{1.4}$$

In fact, this condition (with local integrability) was first considered by Krylov and Röckner in [13], where they proved the existence and uniqueness of strong solutions for the SDE

$$X_{s,t}(x) = x + \int_s^t b(r, X_{s,r}(x)) \, dr + B_t - B_s , \tag{1.5}$$

such that

$$\mathbb{P}\left(\int_0^T |b(t, X_t)|^2 \, dt < \infty\right) = 1 .$$

Similarly, we may consider for convenience the inverse $Y_{s,t} := X_{s,t}^{-1}$, which satisfies the following backward stochastic differential equations,

$$Y_{s,t} = y - \int_s^t b(r, Y_{r,t}) \, dr - (B_t - B_s) , \tag{1.6}$$

for $0 \leq s \leq t$.

The well-posedness of the Cauchy problem (1.3) under condition (1.4) for measurable initial condition was also considered in [22] and [23]. In [3], using a technique based on the regularizing effect observed on expected values of moments of the solution, well-posedness of (1.3) was obtained also for the

limit cases of $p, q = \infty$ or when the inequality in (1.4) becomes an equality. The uniqueness result in that paper are valid for solution in weighed spaces.

We mention that other approaches have also been used to study stochastic linear transport/continuity equations. For example, M. Maurelli in [20] employed the Wiener chaos decomposition to deal with a weakly differentiable drift, in [21], S.A. Mohammed, T.K. Nilssen, and F.N. Proske used Malliavin calculus which allows to deal with just a bounded drift, and in [9] the authors introduced a new class of solutions. We would also like to mention the generalizations to transport-diffusion equations and the associated stochastic differential equations by A. Figalli [10]. and X. Zhang [26].

The main issue of this paper is to prove uniqueness of L^2 -weak solutions for one-dimensional stochastic continuity equation (1.3) with unbounded measurable drift without assumptions on the divergence. More precisely, we assume that b satisfies

$$|b(x)| \leq k(1 + |x|).$$

The proof is based in the fact that one primitive V is regular and verifies the transport equation

$$\partial_t V(t, x) + (b(t, x) + \frac{dB_t}{dt}) \cdot \nabla V(t, x) = 0. \quad (1.7)$$

Then using a modified version of the commutator Lemma and the characteristic systems associated to the SPDE (1.7) we shall show that $V = 0$ with initial condition equal to zero, which implies that $u = 0$.

Other issue in this paper is to give a well-posedness result for solutions in the Sobolev spaces $H^1(\mathbb{R}^d)$ under condition (1.4) with divergence equal to zero. In particular this result implies the persistence of the regularity for initial conditions $u_0 \in H^1(\mathbb{R}^d)$. The proof is based in the Commutator Lemma given by C. Le Bris and P. L. Lions in [17] for solutions with Sobolev regularity. The new result is to show uniqueness in the class of H^1 -solutions, not cover in the previous works (see [8, 3, 22, 23]) with this kind of conditions.

Throughout of this paper, we fix a stochastic basis with a d -dimensional Brownian motion $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, \mathbb{P}, (B_t))$.

2 L^2 - Solutions.

In this section we assume the following hypothesis on vector field and initial condition:

Hypothesis 2.1. *The vector field b satisfies*

$$|b(x)| \leq k(1 + |x|), \quad (2.8)$$

and the initial condition holds

$$u_0 \in L^2(\mathbb{R}, w \, dx) \quad (2.9)$$

where w is the weight defined by $w(x) = e^{2k_2 x^2}$ with $k_2 = 2(k + 99Tk^2)$.

Now, we denote by b^ϵ the standard mollification of b , and let X_t^ϵ be the associated flow given by the SDE (1.5) replacing b by b^ϵ . Similarly, we consider Y_t^ϵ , which satisfies the backward SDE (1.6). We also recall the important results in [24] (see appendix) : let X_t^ϵ be the corresponding stochastic flows, then for all $p \geq 1$ we have that exists constants $C_1 = C_1(k, p, T)$ and $C_2(k, p, T)$ such that

$$\mathbb{E}[|\partial_x X_t^\epsilon(x)|^p] \leq C_1 t^{-\frac{1}{2}} e^{C_2 x^2}, \quad (2.10)$$

the same results is valid for the backward flow Y_t^ϵ since is solution of the same SDE driven by the drifts $-b^\epsilon$. We denote $\mu = (1 + |x|)^2$.

2.1 Definition of solutions

Definition 2.2. *A stochastic process $u \in L^2(\Omega \times [0, T] \times \mathbb{R}, \mu dx)$ is called a L^2 - weak solution of the Cauchy problem (1.3) when: For any $\varphi \in C_0^\infty(\mathbb{R})$, the real valued process $\int u(t, x) \varphi(x) dx$ has a continuous modification which is an \mathcal{F}_t -semimartingale, and for all $t \in [0, T]$, we have \mathbb{P} -almost surely*

$$\begin{aligned} \int_{\mathbb{R}} u(t, x) \varphi(x) dx &= \int_{\mathbb{R}} u_0(x) \varphi(x) \, dx + \int_0^t \int_{\mathbb{R}} u(s, x) b(x) \partial_x \varphi(x) \, dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, x) \, \partial_x \varphi(x) \, dx \odot dB_s. \end{aligned} \quad (2.11)$$

Remark 2.3. *Using the same idea as in Lemma 13 [11], one can write the problem (1.3) in Itô form as follows, a stochastic process $u \in L^2(\Omega \times [0, T] \times \mathbb{R}, \mu dx)$ is a L^2 - weak solution of the SPDE (1.3) iff for every test function $\varphi \in C_0^\infty(\mathbb{R})$, the process $\int u(t, x)\varphi(x)dx$ has a continuous modification which is a \mathcal{F}_t -semimartingale and satisfies the following Itô's formulation*

$$\begin{aligned} \int_{\mathbb{R}} u(t, x)\varphi(x)dx &= \int_{\mathbb{R}} u_0(x)\varphi(x) dx + \int_0^t \int_{\mathbb{R}} u(s, x) b(x) \partial_x \varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x \varphi(x) dx dB_s + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x^2 \varphi(x) dx ds \end{aligned}$$

2.2 Existence.

With the considerations given above we may now prove the existence result

Lemma 2.4. *Assume that hypothesis 2.1 holds. Then there exists L^2 -weak solution of the Cauchy problem (1.3).*

Proof. Step 1: Regularization.

Let $\{\rho_\varepsilon\}_\varepsilon$ be a family of standard symmetric mollifiers and η a nonnegative smooth cut-off function supported on the ball of radius 2 and such that $\eta = 1$ on the ball of radius 1. Now, for every $\varepsilon > 0$, we introduce the rescaled functions $\eta_\varepsilon(\cdot) = \eta(\varepsilon \cdot)$. Thus, we may define the family of regularized coefficients given by

$$b^\varepsilon(x) = \eta_\varepsilon(x)(b * \rho_\varepsilon(x))$$

and

$$u_0^\varepsilon(x) = \eta_\varepsilon(x)(u_0 * \rho_\varepsilon(x)).$$

Clearly we observe that, for every $\varepsilon > 0$, any element b^ε , u_0^ε are smooth (in space) and compactly supported with bounded derivatives of all orders. We observe that to study the stochastic continuity equation (SCE) (1.3) is equivalent to study the stochastic transport equation given by (regularized version):

$$\begin{cases} du^\varepsilon(t, x) + \nabla u^\varepsilon(t, x) \cdot (b^\varepsilon(x)dt + \circ dB_t) + \operatorname{div} b^\varepsilon(x) u^\varepsilon(t, x)dt = 0, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon \end{cases} \quad (2.12)$$

Following the classical theory of H. Kunita [14, Theorem 6.1.9] we obtain that

$$u^\varepsilon(t, x) = u_0^\varepsilon(\psi_t^\varepsilon(x)) \exp \left\{ - \int_0^t \operatorname{div} b^\varepsilon(\phi_s^\varepsilon(\psi_t^\varepsilon(x))) ds \right\}$$

is the unique solution to the regularized equation (2.12), where ϕ_t^ε satisfies the stochastic differential equation (SDE):

$$dX_t = b^\varepsilon(X_t) dt + dB_t, \quad X_0 = x.$$

and ψ_t^ε is the inverse of ϕ_t^ε .

Step 2: Boundedness. Making the change of variables $y = \psi_t^\varepsilon(x) = (\phi_t^\varepsilon(x))^{-1}$ we have that

$$\begin{aligned} \int_{\mathbb{R}} \mathbb{E}[|u^\varepsilon(t, x)|^2] (1 + |x|)^2 dx &= \int_{\Omega} \int_{\mathbb{R}} |u_0^\varepsilon(y)|^2 \exp \left\{ - 2 \int_0^t \operatorname{div} b^\varepsilon(\phi_s^\varepsilon(y)) ds \right\} \times \\ &\quad \times \frac{d\phi_t^\varepsilon(y)}{dy} (1 + |\phi_t^\varepsilon(y)|)^2 dy \mathbb{P}(d\omega) \end{aligned}$$

Now, if we make a minor modification in the proof of the Lemma 3.6 of [24] (see appendix) we obtain that there exist constants $k_1 = k_1(k, T)$ and $k_2 = 2(k + 99Tk^2)$ such that

$$\mathbb{E} \left[\left| \frac{d}{dx} \phi_t^\varepsilon(x) \right|^{-2} \right] = \mathbb{E} \left[\exp \left\{ - 2 \int_0^t \operatorname{div} b^\varepsilon(\phi_s^\varepsilon(x)) ds \right\} \right] \leq k_1 t^{-3/8} e^{k_2 x^2}. \quad (2.13)$$

We also observe that

$$\mathbb{E} \left[|\phi_t^\varepsilon(x)|^4 \right] \leq C(|x|^4 + T^4) \quad (2.14)$$

Then we obtain

$$\begin{aligned} &\mathbb{E} \left[\left| \frac{d}{dx} \phi_t^\varepsilon(x) \right|^{-1} (1 + |\phi_t^\varepsilon(x)|)^2 \right] \\ &\leq C \left(\mathbb{E} \left| \frac{d}{dx} \phi_t^\varepsilon(x) \right|^{-2} + \mathbb{E} (1 + |\phi_t^\varepsilon(x)|)^4 \right) \leq C(k_1 t^{-3/8} e^{k_2 x^2} + T^4 + x^4) \end{aligned}$$

Thus we deduce

$$\begin{aligned}
\int_{\mathbb{R}} \mathbb{E}[|u^\varepsilon(t, x)|^2](1 + |x|)^2 dx &\leq \int_{\mathbb{R}} |u_0^\varepsilon(y)|^2 \mathbb{E} \left[\left| \frac{d\phi_s^\varepsilon(y)}{dy} \right|^{-1} (1 + |\phi_t^\varepsilon(y)|)^2 \right] dy \\
&\leq C \int_{\mathbb{R}} |u_0^\varepsilon(y)|^2 (k_1 t^{-3/8} e^{k_2 x^2} + T^4 + y^4) dy \\
&\leq C k_1 t^{-3/8} \int_{\mathbb{R}} |u_0^\varepsilon(y)|^2 e^{k_2 y^2} dy + C \int_{\mathbb{R}} |u_0^\varepsilon(y)|^2 e^{k_2 y^2} dy.
\end{aligned} \tag{2.15}$$

We observe that

$$\begin{aligned}
\int_{\mathbb{R}} |u_0^\varepsilon(y)|^2 e^{k_2 y^2} dy &\leq \int_{\mathbb{R}} \left[e^{k_2 y^2} \left(\int_{\mathbb{R}} \rho_\varepsilon(y - x) |u_0(x)|^2 dx \right) \right] dy \\
&= \int_{\mathbb{R}} \left[|u_0(x)|^2 \left(\int_{B(x, \varepsilon)} \rho_\varepsilon(y - x) e^{k_2 y^2} dy \right) \right] dx \\
&= \int_{\mathbb{R}} \left[|u_0(x)|^2 \left(\int_{B(0, \varepsilon)} \rho_\varepsilon(u) e^{k_2 (x+u)^2} du \right) \right] dx \\
&\leq \int_{\mathbb{R}} \left[|u_0(x)|^2 e^{2k_2 x^2} \left(\int_{B(0, \varepsilon)} \rho_\varepsilon(u) e^{2k_2 u^2} du \right) \right] dx \\
&\leq C \|u_0\|_{L^2(\mathbb{R}, w dx)}^2
\end{aligned} \tag{2.16}$$

From (2.15) and (2.16) we conclude that

$$\|u^\varepsilon\|_{L^2(\Omega \times [0, T] \times \mathbb{R}, \mu dx)}^2 \leq C(k, T) \|u_0\|_{L^2(\mathbb{R}, w dx)}^2.$$

Therefore, the sequence $\{u^\varepsilon\}_{\varepsilon > 0}$ is bounded in $L^2(\Omega \times [0, T] \times \mathbb{R}, \mu dx)$. Then there exists a convergent subsequence, which we denote also by u^ε , such that converge weakly in $L^2(\Omega \times [0, T] \times \mathbb{R}, \mu dx)$ to some process $u \in L^2(\Omega \times [0, T] \times \mathbb{R}, \mu dx)$.

Step 3: Passing to the Limit. Now, if u^ε is a solution of (2.12), it is also a weak solution, that is, for any test function $\varphi \in C_0^\infty(\mathbb{R})$, u^ε satisfies (written in the Itô form):

$$\begin{aligned}
\int_{\mathbb{R}} u^\varepsilon(t, x) \varphi(x) dx &= \int_{\mathbb{R}} u_0^\varepsilon(x) \varphi(x) dx + \int_0^t \int_{\mathbb{R}} u^\varepsilon(s, x) b^\varepsilon(x) \partial_x \varphi(x) dx ds \\
&\quad + \int_0^t \int_{\mathbb{R}} u^\varepsilon(s, x) \partial_x \varphi(x) dx dB_s + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u^\varepsilon(s, x) \partial_x^2 \varphi(x) dx ds.
\end{aligned}$$

Thus, for prove existence of the SCE (1.3) is enough to pass to the limit in the above equation along the convergent subsequence found. This is made through of the same arguments of [11, theorem 15].

□

2.3 Uniqueness.

Theorem 2.5. *Under the conditions of hypothesis 2.1, uniqueness holds for L^2 - weak solutions of the Cauchy problem (1.3) in the following sense: if u, v are L^2 - weak solutions with the same initial data $u_0 \in L^2(\mathbb{R}, w dx)$, then $u = v$ almost everywhere in $\Omega \times [0, T] \times \mathbb{R}$.*

Proof. Step 0: Set of solutions. Remark that the set of L^2 - weak solutions is a linear subspace of $L^2(\Omega \times [0, T] \times \mathbb{R}, \mu dx)$, because the stochastic continuity equation is linear, and the regularity conditions is a linear constraint. Therefore, it is enough to show that a L^2 - weak solution u with initial condition $u_0 = 0$ vanishes identically.

Step 1: Primitive of the solution. We define $V(t, x) = \int_{-\infty}^x u(t, y) dy$ and we observe that $\partial_x V(t, x)$ belong to $L^2(\Omega \times [0, T] \times \mathbb{R}, \mu dx)$. We consider a nonnegative smooth cut-off function η supported on the ball of radius 2 and such that $\eta = 1$ on the ball of radius 1. For any $R > 0$, we introduce the rescaled functions $\eta_R(\cdot) = \eta(\frac{\cdot}{R})$. Let be $\varphi \in C_0^\infty(\mathbb{R})$, we observe that

$$\int_{\mathbb{R}} V(t, x) \varphi(x) \eta_R(x) dx = - \int_{\mathbb{R}} u(t, x) \theta(x) \eta_R(x) dx - \int_{\mathbb{R}} V(t, x) \theta(x) \partial_x \eta_R(x) dx,$$

where $\theta(x) = \int_{-\infty}^x \varphi(y) dy$. By definition of the solution u , taking as test function $\theta(x) \eta_R(x)$ we have that $V(t, x)$ verifies

$$\begin{aligned} \int_{\mathbb{R}} V(t, x) \eta_R(x) \varphi(x) dx &= - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) b(x) \eta_R(x) \varphi(x) dx ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \eta_R(x) \varphi(x) dx \circ dB_s - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) b(x) \partial_x \eta_R(x) \theta(x) dx ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \partial_x \eta_R(x) \theta(x) dx \circ dB_s - \int_{\mathbb{R}} V(t, x) \theta(x) \partial_x \eta_R(x) dx. \end{aligned} \tag{2.17}$$

Taking the limit as $R \rightarrow \infty$ we obtain

$$\begin{aligned} & \int_{\mathbb{R}} V(t, x) \varphi(x) dx = \\ & - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) b(x) \varphi(x) dx ds - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \varphi(x) dx \circ dB_s. \end{aligned} \quad (2.18)$$

Step 2: Smoothing. Let $\{\rho_\varepsilon(x)\}_\varepsilon$ be a family of standard symmetric mollifiers. For any $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we can use $\rho_\varepsilon(x - \cdot)$ as test function, we get

$$\begin{aligned} \int_{\mathbb{R}} V(t, y) \rho_\varepsilon(x - y) dy &= - \int_0^t \int_{\mathbb{R}} (b(y) \partial_y V(s, y)) \rho_\varepsilon(x - y) dy ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \partial_y V(s, y) \rho_\varepsilon(x - y) dy \circ dB_s \end{aligned}$$

We set $V_\varepsilon(t, x) = (V * \rho_\varepsilon)(x)$, $b_\varepsilon(x) = (b * \rho_\varepsilon)(x)$ and $(bV)_\varepsilon(t, x) = (b \cdot V * \rho_\varepsilon)(x)$. Then we have

$$\begin{aligned} V_\varepsilon(t, x) + \int_0^t b_\varepsilon(x) \partial_x V_\varepsilon(s, x) ds + \int_0^t \partial_x V_\varepsilon(s, x) \circ dB_s \\ = \int_0^t (\mathcal{R}_\varepsilon(V, b))(x, s) ds, \end{aligned}$$

where we denote $\mathcal{R}_\varepsilon(V, b) = b_\varepsilon \partial_x V_\varepsilon - (b \partial_x V)_\varepsilon$.

Step 3: Method of Characteristic. Applying the Itô-Wentzell-Kunita Formula to $V_\varepsilon(t, X_t^\varepsilon)$, see Theorem 8.3 of [15], we have

$$V_\varepsilon(t, X_t^\varepsilon) = \int_0^t (\mathcal{R}_\varepsilon(V, b))(X_s^\varepsilon, s) ds.$$

Then, considering that $X_t^\varepsilon = X_{0,t}^\varepsilon$ and $Y_t^\varepsilon = Y_{0,t}^\varepsilon = (X_{0,t}^\varepsilon)^{-1}$ we see that

$$V_\varepsilon(t, x) = \int_0^t (\mathcal{R}_\varepsilon(V, b))(X_{0,t}^\varepsilon(Y_{0,t}^\varepsilon), s) ds = \int_0^t (\mathcal{R}_\varepsilon(V, b))(Y_{s,t}^\varepsilon, s) ds.$$

Multiplying by the test functions φ and integrating in \mathbb{R} we get

$$\int V_\varepsilon(t, x) \varphi(x) dx = \int_0^t \int (\mathcal{R}_\varepsilon(V, b))(Y_{s,t}^\varepsilon, s) \varphi(x) dx ds. \quad (2.19)$$

We observe that

$$\int_0^t \int (\mathcal{R}_\varepsilon(V, b))(Y_{s,t}^\varepsilon, s) \varphi(x) dx ds = \int_0^t \int (\mathcal{R}_\varepsilon(V, b))(x, s) JX_{s,t}^\varepsilon \varphi(X_{s,t}^\varepsilon) dx ds. \quad (2.20)$$

Step 4: Convergence of the commutator. Now, we observe that $\mathcal{R}_\varepsilon(V, b)$ converge to zero in $L^2([0, T] \times \mathbb{R})$. In fact,

$$(b \partial_x V)_\varepsilon \rightarrow b \partial_x V \text{ in } L^2([0, T] \times \mathbb{R})$$

and by the dominated convergence theorem we obtain

$$b_\varepsilon \partial_x V_\varepsilon \rightarrow b \partial_x V \text{ in } L^2([0, T] \times \mathbb{R}).$$

Step 5: Conclusion. From step 3 we obtain

$$\int V_\varepsilon(t, x) \varphi(x) dx = \int_0^t \int (\mathcal{R}_\varepsilon(V, b))(x, s) JX_{s,t}^\varepsilon \varphi(X_{s,t}^\varepsilon) dx ds, \quad (2.21)$$

Using by Hölder's inequality we have

$$\begin{aligned} & \mathbb{E} \left| \int_0^t \int (\mathcal{R}_\varepsilon(V, b))(x, s) JX_{s,t}^\varepsilon \varphi(X_{s,t}^\varepsilon) dx ds \right| \\ & \leq \left(\mathbb{E} \int_0^t \int |(\mathcal{R}_\varepsilon(V, b))(x, s)|^2 dx ds \right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^t \int |JX_{s,t}^\varepsilon \varphi(X_{s,t}^\varepsilon)|^2 dx ds \right)^{\frac{1}{2}} \end{aligned}$$

From step 4 we have

$$\left(\mathbb{E} \int_0^t \int |(\mathcal{R}_\varepsilon(V, b))(x, s)|^2 dx ds \right)^{\frac{1}{2}} \rightarrow 0.$$

From formula (2.10) we obtain

$$\left(\mathbb{E} \int_0^t \int |JX_{s,t}^\epsilon \varphi(X_{s,t}^\epsilon)|^2 dx ds \right)^{\frac{1}{2}} \leq C \left(\mathbb{E} \int_0^t |\varphi(x)|^2 dx ds \right)^{\frac{1}{2}},$$

Passing to the limit in equation (2.21) we conclude that $V = 0$. Then we deduce that $u = 0$. □

3 $H^1(\mathbb{R}^d)$ Solutions.

We will be considered the divergence-free condition, that is

$$\operatorname{div} b = 0 \tag{3.22}$$

(understood in the sense of distributions).

Definition 3.1. *A stochastic process $u \in L^2(\Omega \times [0, T], H^1(\mathbb{R}^d)) \cap L^\infty(\Omega \times [0, T] \times \mathbb{R}^d)$ is called a H^1 - weak solution of the Cauchy problem (1.3) when: For any $\varphi \in C_0^\infty(\mathbb{R}^d)$, the real valued process $\int u(t, x)\varphi(x)dx$ has a continuous modification which is an \mathcal{F}_t -semimartingale, and for all $t \in [0, T]$, we have \mathbb{P} -almost surely*

$$\begin{aligned} \int_{\mathbb{R}^d} u(t, x)\varphi(x)dx &= \int_{\mathbb{R}^d} u_0(x)\varphi(x) dx - \int_0^t \int_{\mathbb{R}^d} \partial_i u(s, x) \cdot b^i(s, x)\varphi(x) dx ds \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \partial_i u(s, x) \varphi(x) dx \circ dB_s^i. \end{aligned} \tag{3.23}$$

Remark 3.2. *Analogously, as it was done in the remark 2.3 we can write the problem (1.3) in the Itô form as follows, a stochastic process $u \in L^2(\Omega \times [0, T], H^1(\mathbb{R}^d)) \cap L^\infty(\Omega \times [0, T] \times \mathbb{R}^d)$ is a H^1 - weak solution of the SPDE (1.3) iff for every test function $\varphi \in C_0^\infty(\mathbb{R}^d)$, the process $\int u(t, x)\varphi(x)dx$ has a continuous modification which is a \mathcal{F}_t -semimartingale and satisfies the following Itô's formulation*

$$\begin{aligned}
\int u(t, x) \varphi(x) dx &= \int u_0(x) \varphi(x) dx \\
&\quad - \int_0^t \int b^i(s, x) \cdot \varphi(x) \partial_i u(s, x) dx ds \\
&\quad - \int_0^t \int \varphi(x) \partial_i u(s, x) dx dB_s^i \\
&\quad + \frac{1}{2} \int_0^t \int \Delta \varphi(x) u(s, x) dx ds
\end{aligned}$$

3.1 Existence.

Lemma 3.3. *We assume that $u_0 \in H^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and conditions (1.4) and (3.22). Then there exists H^1 -weak solutions u of the Cauchy problem (1.3).*

Proof. Let $\{\rho_\varepsilon\}_\varepsilon$ be a family of standard symmetric mollifiers. Consider a nonnegative smooth cut-off function η supported on the ball of radius 2 and such that $\eta = 1$ on the ball of radius 1. For every $\varepsilon > 0$ introduce the rescaled functions $\eta_\varepsilon(\cdot) = \eta(\varepsilon \cdot)$. Using these two families of functions we define the family of regularized coefficients as $b^\varepsilon(t, x) = \eta_\varepsilon(x) ([b(t, \cdot) * \rho_\varepsilon(\cdot)](x))$. Similarly, define the family of regular approximations of the initial condition $u_0^\varepsilon(x) = \eta_\varepsilon(x) ([u_0(\cdot) * \rho_\varepsilon(\cdot)](x))$.

Remark that any element $b^\varepsilon, u_0^\varepsilon, \varepsilon > 0$ of the two families we have defined is smooth (in space) and compactly supported, therefore with bounded derivatives of all orders. Then, for any fixed $\varepsilon > 0$, the classical theory of Kunita, see [14] or [16], provides the existence of a unique solution u^ε to the regularized equation

$$\begin{cases} du^\varepsilon(t, x, \omega) + \nabla u^\varepsilon(t, x, \omega) \cdot (b^\varepsilon(t, x) dt + \circ dB_t(\omega)) = 0, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon \end{cases} \quad (3.24)$$

together with the representation formula

$$u^\varepsilon(t, x) = u_0^\varepsilon((\phi_t^\varepsilon)^{-1}(x)) \quad (3.25)$$

in terms of the (regularized) initial condition and the inverse flow $(\phi_t^\varepsilon)^{-1}$ associated to the equation of characteristics of (3.24), which reads

$$dX_t = b^\varepsilon(t, X_t) dt + dB_t, \quad X_0 = x.$$

Now, by Lemma 5 of [8] we have that for every $p \geq 1$, there exists $C_{d,p,T} > 0$ such that

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \mathbb{E}[|D(\phi_t^\varepsilon)|^p] \leq C_{d,p,T}, \quad \text{uniformly in } \varepsilon > 0. \quad (3.26)$$

Then, we can use the random change of variables $(\phi_t^\varepsilon)^{-1}(x) \mapsto x$ to obtain that

$$\int_{\mathbb{R}^d} \mathbb{E}|u^\varepsilon(t, x)|^2 dx = \int_{\mathbb{R}^d} \mathbb{E}|u_0^\varepsilon((\phi_t^\varepsilon)^{-1}(x, \omega))|^2 dx = \int_{\mathbb{R}^d} |u_0^\varepsilon(x)|^2 dx \quad (3.27)$$

Moreover, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \mathbb{E}|\nabla u^\varepsilon(t, x)|^2 dx &= \int_{\mathbb{R}^d} \mathbb{E}|\nabla[u_0^\varepsilon((\phi_t^\varepsilon)^{-1}(x, \omega))]|^2 dx \\ &= \int_{\mathbb{R}^d} |\nabla u_0^\varepsilon((\phi_t^\varepsilon)^{-1}) D(\phi_t^\varepsilon)^{-1}(x, \omega)|^2 dx \\ &= \int_{\mathbb{R}^d} |\nabla u_0^\varepsilon(x)|^2 \mathbb{E}|D(\phi_t^\varepsilon)^{-1}(\phi_t^\varepsilon, \omega)|^2 dx \end{aligned} \quad (3.28)$$

We observe that

$$D(\phi_t^\varepsilon)^{-1}(\phi_t^\varepsilon, \omega) = D^{-1}(\phi_t^\varepsilon)$$

and

$$D^{-1}(\phi_t^\varepsilon) = \text{Cof}(D\phi_t^\varepsilon)^T$$

where Cof denoted the cofactor matrix of $D\phi_t^\varepsilon$. By inequality (3.26) we deduce that $\text{Cof}(D\phi_t^\varepsilon)^T \in L^\infty(\mathbb{R}^d \times [0, T], L^2(\Omega))$. Thus we obtain that

$$\begin{aligned} \int_{\mathbb{R}^d} \mathbb{E}|\nabla u^\varepsilon(t, x)|^2 dx &= \int_{\mathbb{R}^d} |\nabla u_0^\varepsilon(x)|^2 \mathbb{E}|D(\phi_t^\varepsilon)^{-1}(\phi_t^\varepsilon, \omega)|^2 dx \\ &\leq C \int_{\mathbb{R}^d} |\nabla u_0^\varepsilon(x)|^2 dx \end{aligned} \quad (3.29)$$

If u^ε is a solution of (3.24), it is also a weak solution, which means that for any test function $\varphi \in C_0^\infty(\mathbb{R}^d)$, u^ε satisfies the following equation (written in Itô form)

$$\begin{aligned} \int_{\mathbb{R}^d} u^\varepsilon(t, x) \varphi(x) dx &= \int_{\mathbb{R}^d} u_0^\varepsilon(x) \varphi(x) dx - \int_0^t \int_{\mathbb{R}^d} \partial_i u^\varepsilon(s, x) b^{i, \varepsilon}(s, x) \varphi(x) dx ds \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \partial_i u^\varepsilon(s, x) \varphi(x) dx dB_s^i + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} u^\varepsilon(s, x) \Delta \varphi(x) dx ds. \end{aligned} \quad (3.30)$$

To prove the existence of weak solutions to (1.3) we can extract subsequence (for simplicity the whole sequence), which converges weakly to some u in $L^2(\Omega \times [0, T], H^1(\mathbb{R}^d))$ and weak star in $L^\infty(\Omega \times [0, T] \times \mathbb{R}^d)$. Then via classical arguments, that is, we can pass to the limit in the above equation along this subsequence. This is done following the classical argument of [25, Sect. II, Chapter 3], [11, Theorem 15] and [3, Theorem 23]. \square

3.2 Uniqueness.

Before starting and proving the main theorem of this subsection, we shall introduce some further notation and the key lemma on commutators.

Let $\{\rho_\varepsilon\}$ be a family of standard positive symmetric mollifiers. Given two functions $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $g : \mathbb{R}^d \mapsto \mathbb{R}$, the commutator $\mathcal{R}_\varepsilon(f, g)$ is defined as

$$\mathcal{R}_\varepsilon(f, g) := (f \cdot \nabla)(\rho_\varepsilon * g) - \rho_\varepsilon * (f \cdot \nabla g). \quad (3.31)$$

The following lemma is due to C. Le Bris and P.-L. Lions [17, Lemma 5.1].

Lemma 3.4. (*C. Le Bris - P. L. Lions*) *Let $f \in L_{loc}^2(\mathbb{R}^d)$, $g \in H^1(\mathbb{R}^d)$. Then, passing to the limit as $\varepsilon \rightarrow 0$*

$$\mathcal{R}_\varepsilon(f, g) \rightarrow 0 \quad \text{in} \quad L_{loc}^1(\mathbb{R}^d).$$

We can finally state our uniqueness result.

Theorem 3.5. *Under the conditions (1.4) and (3.22), uniqueness holds for H^1 - weak solutions of the Cauchy problem (1.3) in the following sense: if u, v are H^1 - weak solutions with the same initial data $u_0 \in H^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then $u = v$ almost everywhere in $\Omega \times [0, T] \times \mathbb{R}^d$.*

Proof. The proof is essentially based on energy-type estimates on u . However, to rigorously obtain it two preliminary technical steps of regularization and localization are needed, where the above Lemma 3.4 will be used to deal with the commutators appearing in the regularization process.

Step 0: Set of solutions. Remark that the set of H^1 - weak solutions is a linear subspace of $L^2(\Omega \times [0, T], H^1(\mathbb{R}^d)) \cap L^\infty(\Omega \times [0, T] \times \mathbb{R}^d)$, because the stochastic transport equation is linear, and the regularity conditions is a linear constraint. Therefore, it is enough to show that a H^1 - weak solution u with initial condition $u_0 = 0$ vanishes identically.

Step 1: Smoothing. Let $\{\rho_\varepsilon(x)\}_\varepsilon$ be a family of standard symmetric mollifiers. For any $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we can use $\rho_\varepsilon(x - \cdot)$ as test function, we get

$$\begin{aligned} \int_{\mathbb{R}^d} u(t, y) \rho_\varepsilon(x - y) dy &= - \int_0^t \int_{\mathbb{R}^d} (b^i(s, y) \cdot \partial_i u(s, y)) \rho_\varepsilon(x - y) dy ds \\ &\quad - \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \partial_i u(s, y) \partial_i \rho_\varepsilon(x - y) dy ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} u(s, y) \partial_i \rho_\varepsilon(x - y) dy dB_s^i \end{aligned}$$

Set $u_\varepsilon(t, x) = u(t, x) * \rho_\varepsilon(x)$. Using the definition (3.31) of the commutator $(\mathcal{R}_\varepsilon(f, g))(s)$ with $f = b(s, \cdot)$ and $g = u(s, \cdot)$, we have for each $t \in [0, T]$

$$\begin{aligned} u_\varepsilon(t, x) &+ \int_0^t b^i(s, x) \cdot \partial_i u_\varepsilon(s, x) ds - \frac{1}{2} \int_0^t \Delta u_\varepsilon(s, x) ds \\ &\quad + \int_0^t \partial_i u_\varepsilon(s, x) dB_s^i \\ &= \int_0^t (\mathcal{R}_\varepsilon(b, u))(s) ds. \end{aligned}$$

By Itô formula we have

$$\begin{aligned} u_\varepsilon^2(t, x) &= - \int_0^t b^i(s, x) \cdot \partial_i u_\varepsilon^2(s, x) ds - \int_0^t \int_{\mathbb{R}^d} \partial_i u_\varepsilon^2(s, x) dx dB_s^i \\ &\quad + \frac{1}{2} \int_0^t \Delta u_\varepsilon^2(s, x) dx + \int_0^t 2u_\varepsilon(s, x) \mathcal{R}_\varepsilon(b, u)(s, x) ds \end{aligned} \quad (3.32)$$

Step 2: Localization. Consider a nonnegative smooth cut-off function η supported on the ball of radius 2 and such that $\eta = 1$ on the ball of radius 1. For any $R > 0$ introduce the rescaled functions $\eta_R(\cdot) = \eta(\frac{\cdot}{R})$. Multiplying (3.32) by η_R and integrating over \mathbb{R}^d we have

$$\begin{aligned} \int_{\mathbb{R}^d} u_\epsilon^2(t, x) \eta_R(x) dx &= - \int_0^t \int_{\mathbb{R}^d} b^i(s, x) \cdot \partial_i u_\epsilon^2(s, x) \eta_R(x) dx ds \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \partial_i u_\epsilon^2(s, x) \eta_R(x) dx dB_s^i \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \Delta u_\epsilon^2(s, x) \eta_R(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} 2u_\epsilon(s, x) \mathcal{R}_\epsilon(b, u)(s, x) \eta_R(x) ds \end{aligned}$$

which we rewrite as

$$\begin{aligned} \int_{\mathbb{R}^d} u_\epsilon^2(t, x) \eta_R(x) dx &= \int_0^t \int_{\mathbb{R}^d} u_\epsilon^2(s, x) b^i(s, x) \cdot \partial_i \eta_R(x) dx ds \\ &\quad + \int_0^t \left(\int_{\mathbb{R}^d} u_\epsilon^2(s, x) \partial_i \eta_R(x) dx \right) dB_s^i \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} u_\epsilon^2(s, x) \Delta \eta_R(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} 2u_\epsilon(s, x) \mathcal{R}_\epsilon(b, u)(s, x) \eta_R(x) ds \quad (3.33) \end{aligned}$$

Step 4: Passage to the limit. Finally, in this step we shall pass to the limit in ϵ and R to obtain uniqueness. We first take the limit $\epsilon \rightarrow 0$ in the above equation (3.33). By standard properties of mollifiers $u_\epsilon \rightarrow u$ strongly

in $L^2([0, T]; H^1(\mathbb{R}^d))$. Then using Lemma 3.4, we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} u^2(t, x) \eta_R(x) dx &= \int_0^t \int_{\mathbb{R}^d} u^2(s, x) b^i(s, x) \cdot \partial_i \eta_R(x) dx ds \\ &\quad + \int_0^t \left(\int_{\mathbb{R}^d} u^2(s, x) \partial_i \eta_R(x) dx \right) dB_s^i \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} u^2(s, x) \Delta \eta_R(x) dx ds. \end{aligned} \quad (3.34)$$

Now, we observe that

$$\left| \int_0^t \int_{\mathbb{R}^d} u^2(s, x) b^i(s, x) \cdot \partial_i \eta_R(x) dx ds \right| \leq C \left| \int_0^t \left(\int_{R < |x| < 2R} |b(s, x)|^p dx \right)^{q/p} ds \right|^{1/q}.$$

Thus, taking the limit in (3.34) as $R \rightarrow \infty$ we have

$$\int_{\mathbb{R}^d} u^2(t, x) dx = 0$$

Taking expectation and integrating on $[0, T]$ we have

$$\int_{\Omega} \int_0^T \int_{\mathbb{R}^d} u^2(t, x) dx dt \mathbb{P}(d\omega) = 0.$$

Therefore, we conclude that $u = 0$ almost everywhere on $\Omega \times [0, T] \times \mathbb{R}^d$. \square

4 Appendix

Lemma 4.1. *Assume $b \in C_c^\infty(\mathbb{R})$ and satisfy the hypothesis 2.1. Then we may choose $T > 0$ such that there exists a constants $k_1 = k_1(k, T)$ and $k_2 = k_2(k, T)$ such that*

$$\mathbb{E} \left[\left| \frac{d}{dx} X_t(x) \right|^{-2} \right] \leq k_1 t^{-3/8} e^{k_2 x^2}, \quad (4.35)$$

where $k_1 = \sqrt{c_1} \sqrt[4]{c_2} e^{35Tk^2}$ and $k_2 = 2(k + 99Tk^2)$ (c_1 and c_2 are defined below in the proof).

Proof. We have the SDE associated to the vector field b :

$$dX_t = b(X_t) dt + dB_t, \quad X_0 = x.$$

We denote

$$\mathcal{E} \left(\int_0^t b(X_u) dB_u \right) = \exp \left\{ \int_0^t b(X_u) dB_u - \frac{1}{2} \int_0^t b^2(X_u) du \right\}$$

and

$$dQ(\omega) = \mathcal{E} \left(\int_0^t b(X_u) dB_u \right) d\mathbb{P}(\omega).$$

Using of Girsanov's theorem we obtain that

$$\begin{aligned} \mathbb{E} \left[\left| \frac{dX_t}{dx}(x) \right|^{-2} \right] &= \mathbb{E}_Q \left[\left| \frac{dY_t}{dx}(x) \right|^{-2} \right] \\ &= \mathbb{E} \left[\exp \left\{ -2 \int_0^t b'(x + B_s) ds \right\} \mathcal{E} \left(\int_0^t b(x + B_s) dB_s \right) \right] \end{aligned}$$

Now, we proceed as in the proof of the Lemma 3.6 of [24]. Let $b_1 = -b$, then we have

$$\mathbb{E} \left[\left| \frac{dX_t}{dx}(x) \right|^{-2} \right] = \mathbb{E} \left[\exp \left\{ 2 \int_0^t b'_1(x + B_s) ds \right\} \mathcal{E} \left(\int_0^t b(x + B_s) dB_s \right) \right]$$

By the Itô's formula to $\tilde{b}(z) = \int_\infty^z b_1(y) dy$ result

$$\tilde{b}(x + B_t) = \tilde{b}(x) + \int_0^t b_1(x + B_s) dB_s + \frac{1}{2} \int_0^t b'_1(x + B_s) ds.$$

Applying the Hölder inequality we get

$$\begin{aligned} \mathbb{E} \left[\left| \frac{dX_t}{dx}(x) \right|^{-2} \right] &= \mathbb{E} \left[\exp \left\{ 4(\tilde{b}(x + B_t) - \tilde{b}(x) - \int_0^t b_1(x + B_s) dB_s) \right\} \mathcal{E} \left(\int_0^t b(x + B_s) dB_s \right) \right] \\ &\leq \| \exp \{ 4(\tilde{b}(x + B_t) - \tilde{b}(x)) \} \|_{L^2(\Omega)} \| \exp \left\{ -4 \int_0^t b_1(x + B_s) dB_s \right\} \|_{L^2(\Omega)} \\ &\quad \times \mathbb{E} \left(\int_0^t b(x + B_s) dB_s \right) \|_{L^2(\Omega)}. \end{aligned} \tag{4.36}$$

For the first term, we have

$$\begin{aligned}
|\tilde{b}(x + B_t) - \tilde{b}(x)| &= \left| \int_0^1 b_1(x + \theta(B_t)) d\theta \right| |B_t| \\
&\leq \int_0^1 (k + k|x + \theta B_t|) d\theta |B_t| \\
&\leq k|B_t| + k|x||B_t| + \frac{k}{2}(B_t)^2 \\
&\leq \frac{k}{2}x^2 + k|B_t| + k(B_t)^2
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}[\exp\{8(\tilde{b}(x + B_t) - \tilde{b}(x))\}] &\leq \mathbb{E}[\exp\{8(\frac{k}{2}x^2 + k|B_t| + k(B_t)^2)\}] \\
&= e^{4kx^2} \mathbb{E}[\exp\{8(k|B_t| + k(B_t)^2)\}] \\
&= e^{4kx^2} \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \exp\{8k(|z| + z^2) - \frac{z^2}{2t}\} dz
\end{aligned}$$

Thus, we conclude that

$$\|\exp\{4(\tilde{b}(x + B_t) - \tilde{b}(x))\}\|_{L^2(\Omega)} \leq e^{2kx^2} t^{-1/4} \sqrt{c_1} \quad (4.37)$$

where

$$c_1 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\{8k(|z| + z^2) - \frac{z^2}{2T}\} dz$$

For the second term of (4.36) we have

$$\begin{aligned}
& \mathbb{E} \left[\exp \left\{ -8 \int_0^t b_1(x + B_s) dB_s \right\} \mathcal{E} \left(\int_0^t b(x + B_s) dB_s \right)^2 \right] \\
&= \mathbb{E} \left[\exp \left\{ -8 \int_0^t b_1(x + B_s) dB_s \right\} \exp \left\{ 2 \int_0^t b(x + B_s) dB_s - \int_0^t b^2(x + B_s) ds \right\} \right] \\
&= \mathbb{E} \left[\exp \left\{ -10 \int_0^t b_1(x + B_s) dB_s - \int_0^t b_1^2(x + B_s) ds \right\} \right] \\
&= \mathbb{E} \left[\exp \left\{ -10 \int_0^t b_1(x + B_s) dB_s - \alpha \int_0^t b_1^2(x + B_s) ds \right\} \exp \left\{ (\alpha - 1) \int_0^t b_1^2(x + B_s) ds \right\} \right] \\
&\leq \left\| \exp \left\{ -10 \int_0^t b_1(x + B_s) dB_s - \alpha \int_0^t b_1^2(x + B_s) ds \right\} \right\|_{L^2(\Omega)} \times \\
&\quad \times \left\| \exp \left\{ (\alpha - 1) \int_0^t b_1^2(x + B_s) ds \right\} \right\|_{L^2(\Omega)} \tag{4.38}
\end{aligned}$$

Now, we choose $\alpha = 100$ because $\frac{1}{2}(-20b_1(x + B_s))^2 = 2\alpha b_1^2(x + B_s)$ and in this way the process $\exp\{-20 \int_0^t b_1(x + B_s) dB_s - 200 \int_0^t b_1^2(x + B_s) ds\} = \mathcal{E} \left(\int_0^t (-20b_1(x + B_s) dB_s) \right)$ is a martingale with expectation equal to one. Then

$$\left\| \exp \left\{ -10 \int_0^t b_1(x + B_s) dB_s - 100 \int_0^t b_1^2(x + B_s) ds \right\} \right\|_{L^2(\Omega)} = 1$$

Using (2.8) we obtain that the second term of (4.38) is bounded by

$$\begin{aligned}
\mathbb{E} \left[\exp \left\{ 2(\alpha - 1) \int_0^t b_1^2(x + B_s) ds \right\} \right] &= \mathbb{E} \left[\exp \left\{ 198 \int_0^t b_1^2(x + B_s) ds \right\} \right] \\
&\leq \mathbb{E} \left[\exp \left\{ 198 \int_0^t k^2(1 + |x + B_s|)^2 ds \right\} \right] \\
&\leq \mathbb{E} \left[\exp \left\{ 198tk^2(1 + B_t^*)^2 \right\} \right],
\end{aligned}$$

where $B_t^* = \sup_{s \leq t} |x + B_s|$. We define

$$Y_s = \exp\{99tk^2(1 + |x + B_s|)^2\}$$

Then, by Doob's Maximal inequality we have

$$\begin{aligned}
\mathbb{E} \left[\exp \left\{ 198tk^2(1 + B_t^*)^2 \right\} \right] &= \mathbb{E} \left[\sup_{s \leq t} Y_s^2 \right] \\
&\leq 4 \mathbb{E}[Y_t^2] = \mathbb{E}[\exp\{198tk^2(1 + |x + B_t|)^2\}] \\
&\leq 4 \mathbb{E}[\exp\{396tk^2(1 + (x + B_t)^2)\}] \\
&\leq 4 \mathbb{E}[\exp\{396tk^2(1 + 2(x^2 + B_t^2))\}] \\
&= 4e^{396tk^2} e^{792tk^2x^2} \mathbb{E}[\exp\{792tk^2B_t^2\}] \\
&= 4e^{396tk^2} e^{792tk^2x^2} \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \exp\{792tk^2z^2 - \frac{z^2}{2t}\} dz
\end{aligned}$$

Substituting in (4.38) we get

$$\mathbb{E} \left[\exp \left\{ -8 \int_0^t b_1(x + B_s) dB_s \right\} \mathcal{E} \left(\int_0^t b(x + B_s) dB_s \right)^2 \right] \leq e^{198Tk^2} e^{396Tk^2x^2} t^{-1/4} \sqrt{c_2}, \quad (4.39)$$

where

$$c_2 = \frac{4}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\{792Tk^2z^2 - \frac{z^2}{2T}\} dz$$

Therefore, replacing (4.37) and (4.39) in (4.36) we obtain

$$\begin{aligned}
\mathbb{E} \left[\left| \frac{dX_t}{dx}(x) \right|^{-2} \right] &\leq e^{2kx^2} t^{-1/4} \sqrt{c_1} e^{99Tk^2} e^{198Tk^2x^2} t^{-1/8} \sqrt[4]{c_2} \\
&= \sqrt{c_1} \sqrt[4]{c_2} e^{99Tk^2} t^{-3/8} e^{2(k+99Tk^2)x^2} \\
&= k_1 t^{-3/8} e^{k_2 x^2},
\end{aligned}$$

where $k_1 = \sqrt{c_1} \sqrt[4]{c_2} e^{99Tk^2}$ and $k_2 = 2(k + 99Tk^2)$. This prove (4.35). \square

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